LINEAR VISCOELASTICITY WITH COUPLE STRESSES

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WITH

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ABSTRACT

The effects of Couple Stresses in Viscoelasticity have been considered. Constitutive equations have been proposed for force and Couple Stresses. An attempt has been made towards formulating boundary value problems.

CONTENTS

0. INTRODUCTION

0.1. Viscoelasticity describes the property of a certain class of materials, which exhibit both the properties of viscosity and elasticity. In general the term fluid is used to denote the extreme class in which the elastic property is negligibly small or zero. On the other hand most solids, under normal conditions, show little or no viscous behaviour. Both effects are of importance in materials like metals at high temperatures, rubbers, plastics and most high polymers.

Viscoelasticity theory allows for dissipation, an effect which is not considered by classical elasticity theory. A purely elastic solid can remain in equilibrium under the action of shear. There is a natural state to which the substance returns when stresses are removed, and because of this, the work done in deforming an elastic solid may be entirely recovered by allowing the solid to revert to its natural state. On the other hand a fluid cannot be in equilibrium under shear. It will flow if shearing stresses, however small, are applied. It does not have a natural state and as such work done on it by shear is dissipated as heat.

Purely viscous and purely elastic substances are idealisations; theories based on these idealisations agree with the observed behaviour of matter for only

certain selected materials and under certain conditions. Hence there is a basic need to develop a suitable theory for those materials which do not fall under such catagories.

0.2. The origins of the linear theory of visco-elasticity may be traced to various isolated researches during the second half of the 19th century. According to Gurtin and Sternberg (1), Boltzmann first formulated the three dimensional theory for an isotropic medium in the year of 1874. Later on V. Volterra generalised this theory to the anisotropic case. The subject then remained comparatively dormant. It has in recent years attracted a renewed and growing interest, largly as a consequence of advances in high polymer technology.

Modern theoretical investigations on the subject may roughly be divided into two principal categories, (a) those concerned with the study of the underlying constitutive equations and (b) those pertaining to the solutions of boundary-value problems. The first category includes : alternative reformulation of the one dimensional stress-strain relations and studies of the connection between the corresponding alternative material response characteristics; attempts to predict the form of the constitutive relations on the basis of molecular models; the deduction of the one-dimensional linear heriditary law from a set of macroscopic postulates

thermodynamic studies of the foundation of the linear theory; the re-examination of the linear theory from the view point of non-linear continuum mechanics. The second category of investigations deals with methods of integration appropriate to the fundamental boundary value problems. An example being the correspondence principle which links the linear theories of visco-elasticity and elasticity. Some physical chemists such as Ferry and Tobolsky have been concerned with the linear response of viscoelastic materials with the aim of learning some thing about the molecular constitution of matter. Bland and others have solved a large number of boundary value problems using different formulations. More general formulations have been proposed by mathematicians and those working in the fields of theoretical mechanics.

We will develop a linear theory of visco-elasticity which takes into account the effects of couple stresses.

1. CLASSICAL VISCOELASTICITY

1.1. One dimensional formulation.

A generalised viscoelastic solid is specified by the existence of a functional equation of state connecting stress (σ), strain (ϵ), time (t) and temperature (θ)

$$F(\sigma, \epsilon, t, \theta) = 0 \qquad \ldots 1.1.1$$

Initially, for simplicity, the discussion is restricted to a solid stressed under uniaxial tension or compression. The appropriate generalisation to include three dimensional cases is given later. The relation 1.1.1 which may include time differentials and integral operators of arbitrary orders, immediately excludes problems associated with the slip and dislocation of metals and fracture of solids.

By expanding the functional relation 1.1.1 to the lowest order of non-vanishing terms, (linear in stress and strain) and by assuming the temperature to enter only as a constant parameter, a relation for isothermal behaviour can be derived which conforms to the Boltzmann superposition principle for linear viscoelastic solids.

The physical phenomena, which distinguish viscoelastic substances from pure viscous fluids and pure elastic solids are recovery of strain or recoil

when the stress is released and the flow or creep under constant stress respectively.

Of the many pairs of associated stress-strain cycles, it is usual to choose either creep or relaxation behaviour as a fundamental measure of the mechanical properties of a linear solid. Assuming the response to contain instantaneous components of stress and strain related by a dynamic modulus the creep and relaxation behaviour may be expressed by the linear relation

$$\epsilon(t) = E^{-1}\sigma(t) + \int_{-\infty}^{t} \Psi(t-\tau)\frac{d\sigma(\tau)}{d\tau}d\tau \quad \ldots \; 1.1.2$$

$$\sigma(t) = E\,\epsilon(t) - \int_{-\infty}^{t} \Phi(t-\tau)\frac{d\epsilon(\tau)}{d\tau}d\tau \quad \ldots \; 1.1.3$$

Here $\Psi(t)$ and $\Phi(t)$ are the uniaxial creep and relaxation functions.

$\Psi(t)$ and $\Phi(t)$ are now non-dimensionalised by expressing

$$\psi(t) = E_1\,\Psi(t)$$
$$\varphi(t) = E_1^{-1}\Phi(t) \quad \ldots \; 1.1.4$$

where E_1 is another constant having the dimension as that of E

In terms of non-dimensionalised creep function $\psi(t)$ and relaxation function $\varphi(t)$, equations 1.1.2 and 1.1.3 are written down as :

$$\epsilon(t) = E^{-1}\sigma(t) + E_1^{-1}\int_{-\infty}^{t} \psi(t-\tau)\frac{d\sigma(\tau)}{d\tau}d\tau \quad \ldots \; 1.1.5$$

$$\sigma(t) = E\,\epsilon(t) - E_1\int_{-\infty}^{t} \varphi(t-\tau)\frac{d\epsilon(\tau)}{d\tau}d\tau \quad \ldots \; 1.1.6$$

In writing down 1.1.5 and 1.1.6, it has been assumed that E^{-1} and E_1^{-1} are finite quantities. While this assumption excludes some of the highly idealised models of viscoelasticity (e.g., Voigt solid, Newtonian fluid), it is almost certainly correct for real solids. We will discuss at a later stage how these equations can be used with certain restrictions for Newtonian fluids and Voigt Solids. In the following discussions these catagories should not be included unless specifically mentioned.

1.2. Forms of the functions φ and ψ.

From physical observations we note that during creep the strain increases at a constant value of stress, while during relaxation the stress decreases at a constant value of strain. By considering equations 1.1.5 and 1.1.6 and keeping in mind the types of physical behaviours described above, we conclude that : (i) φ (t) must be a monotonic increasing function of time (Newtonian fluid and Voigt Solid being exceptions).

(ii) ψ (t) must be a monotonic function of time (Newtonian fluid being an exception).

(iii) As σ (t) can never change sign during relaxation, φ (t) is bounded by its upper limit $\varphi(\infty)$ where $\frac{1}{K} \geqslant \phi(\infty)$, K being equal to $\frac{E_1}{E}$.

φ (t) and ψ (t) are related. This can be shown by taking the Laplace transforms of equations

1.1.5 and 1.1.6. The equations obtained are :

$$\bar{\epsilon}(s) = [E^{-1} + E_1^{-1} s \bar{\Psi}(s)]\bar{\sigma}(s) \qquad \text{... 1.2.1}$$

$$\bar{\sigma}(s) = [E - E_1 s \bar{\Phi}(s)]\bar{\epsilon}(s) \qquad \text{... 1.2.2}$$

Equivalence of the above relations yields

$$[E - E_1 s \bar{\Phi}(s)][E^{-1} + E_1^{-1} s \bar{\Psi}(s)] = 1 \qquad \text{... 1.2.3}$$

which in the case of $E = E_1$ reduces to

$$[1 - s \bar{\Phi}(s)][1 + s \bar{\Psi}(s)] = 1 \qquad \text{... 1.2.4}$$

Equation 1.2.4 is due to Gross (2) and is of some importance in co-relating the creep and relaxation behaviours of a typical viscoelastic material.

Making use of the Abelian theorem (for $S \rightarrow 0$, through positive values of S, $\underset{s\rightarrow 0}{Lt}\, s\bar{f}(s) = \underset{t\rightarrow\infty}{Lt}\, f(t)$) and using equation 1.2.3 we get :

$$[E - E_1 \Phi(\infty)][E^{-1} + E_1^{-1} \Psi(\infty)] = 1 \qquad \text{... 1.2.5}$$

which in case of $E = E_1$ reduces to :

$$[1 - \Phi(\infty)][1 + \Psi(\infty)] = 1 \qquad \text{... 1.2.6}$$

From 1.2.5 we get $\Psi(\infty) = \dfrac{K^2 \Phi(\infty)}{1 - K\Phi(\infty)}$ ... 1.2.7

Connecting the ultimate values of the creep and relaxation functions. For a partially relaxing solid $K\Phi(\infty) < 1$ (perspex at ambient temperature) while for a completely relaxing solid (Polyisobutylene at ambient temperature) $K\Phi(\infty) = 1$. So $\Psi(\infty)$ must be bounded for a partially relaxing solid and unbounded for a completely relaxing solid.

1.3. Stress relaxation.

It is of practical interest to formulate the type of φ function symbolising different material behaviours. For solids which exhibit instantaneous elastic behaviour (Maxwell solid), $\varphi(t) = (1-e^{-t})$ is the simplest form for φ obeying all the restrictions discussed earlier.

Substituting in equation 1.1.6 we get

$$\sigma(t) = E\,\epsilon(t) - E_1 \int_{-\infty}^{t} \left(1 - e^{-(t-\tau)}\right) \frac{d\epsilon(\tau)}{d\tau}\, d\tau \quad \ldots 1.3.1$$

It we take $\epsilon(t) = \epsilon_0 H(t-\tau)$, then at $\tau = t$ it gives the instantaneous response as $\sigma = E\,\epsilon_0$.

For $t = \infty$ it gives

$$\sigma(\infty) = E\epsilon_0 - E_1\,\epsilon_0 \quad \ldots 1.3.2$$

Complete relaxation takes place when $E = E_1$ whereas the stress assumes a constant value at $t = \infty$ in case of $E > E_1$. This has been shown in Fig. 1.

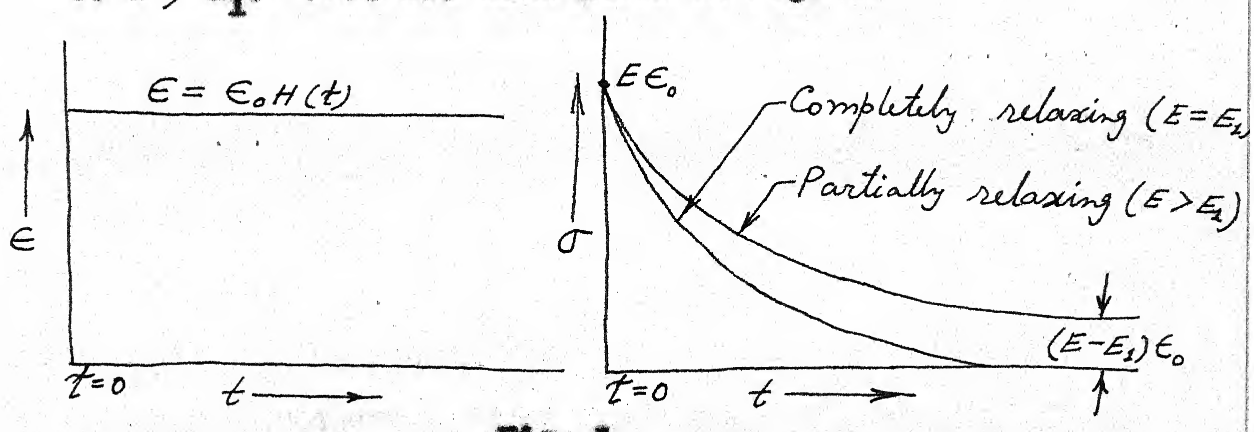


Fig. 1
Complete or Partial relaxation of Maxwell solid

It should be noted that in choosing $\varphi(t) = (1-e^{-t})$, we have imposed a unique relaxation pattern

to all materials. To account for quick relaxation or slow relaxation behaviour the general form proposed is:

$$\varphi(t) = 1 - e^{-\alpha t} \quad \ldots 1.3.3$$

α being a material constant which is positive.

1.4. Creep.

Consider the equation 1.1.5 defining strain at time t

$$\epsilon(t) = E^{-1}\sigma(t) + E_1^{-1}\int_{-\infty}^{t}\psi(t-\tau)\frac{d\sigma(\tau)}{d\tau}d\tau \quad \ldots 1.1.5$$

The general form of ψ (t) is obtained from the relation given (in section 1.2) in the Laplace transform plane as

$$\left[E - E_1 s\bar{\varphi}(s)\right]\left[E^{-1} + E_1^{-1} s\bar{\psi}(s)\right] = 1 \quad \ldots 1.2.3$$

where φ (t) is given by φ(t) $= \left[1 - e^{-\alpha t}\right]$ ... 1.3.3

The class of solids known as Voigt Solids do not exhibit any relaxation phenomena. Their instantaneous response to suddenly applied load is nil, but they show a well defined creep phenomena. Equation 1.1.6 cannot be used for such materials, whereas we can use equation 1.1.5 with E $\approx$ infinity. This gives the strain at any time as :

$$\epsilon(t) = E_1^{-1}\int_{-\infty}^{t}\psi(t-\tau)\frac{d\sigma(\tau)}{d\tau}d\tau \quad \ldots 1.4.1$$

ψ(t) may be chosen keeping in mind the fact that such a solid possesses elastic and viscous elements in parallel in the macroscopic level, and as such creep can never be infinite. Hence the creep function is bounded.

ψ can be taken as :

$$\psi(t) = [1 - e^{-\beta t}] \quad \dots 1.4.2$$

where β is a material constant which is positive. The behaviour of a Voigt solid is shown in Fig. 2.

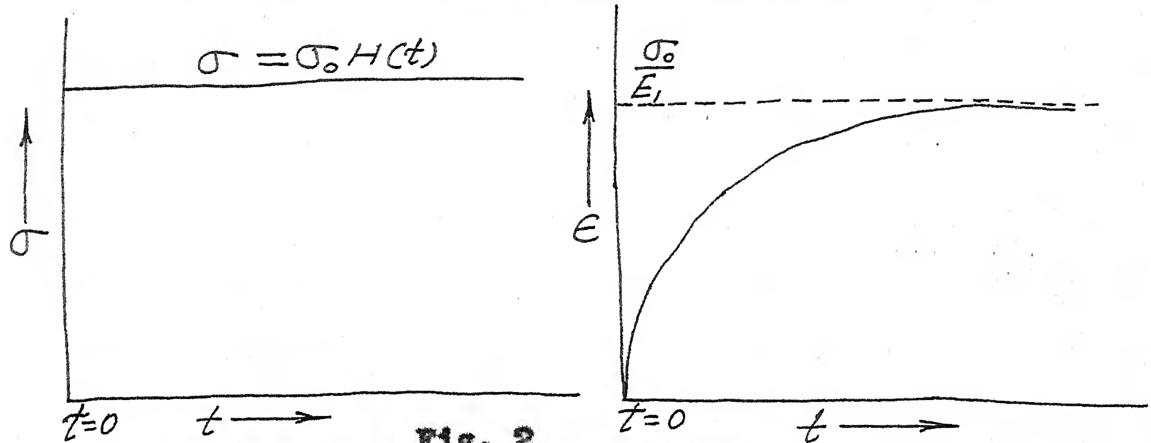


Fig. 2
Creep of a typical Voigt Solid

1.5. Ideal Hookean Solids and Newtonian fluids.

For elastic solids $\varphi(t) = 0$, $\psi(t) = 0$. For this case equations 1.1.5 and 1.1.6 reduce to

$$\epsilon(t) = E^{-1}\sigma(t)$$
$$\sigma(t) = E\,\epsilon(t) \quad \dots 1.5.1$$

giving us the behaviour of Hookean Solids in the uni-axial case.

For a Newtonian fluid the instantaneous elastic part is dropped from equation 1.1.6 to give

$$\sigma(t) = -E_1 \int_{-\infty}^{t} \varphi(t-\tau)\frac{d\epsilon(\tau)}{d\tau}d\tau \quad \dots 1.5.2$$

The suitable form of $\varphi(t) = \delta(t)$ ... 1.5.3

where $\delta(t)$ is the Dirac's Delta function such that $\int_{-\infty}^{t}\delta(t-\tau)\,d\tau = 1$, reduces equation 1.5.2 to

$$\sigma(t) = -E_1 \frac{d\epsilon(t)}{dt} \quad \dots 1.5.4$$

The negative sign can be interpreted by taking $E_1 \frac{d \in(t)}{dt}$ as the internal resistance acting opposite to the applied traction. It should be noted that $\sigma(t)$ in equation 1.5.4 is the shear stress and E_1 has the dimension of viscosity $(\frac{M}{TL})$.

1.6. Generalisation to three dimensions.

The constitutive equations for the three dimensional case will now be considered. It will be assumed that the material is isotropic in behaviour. As an isotropic elastic solid is specified by two independent elastic constants, the behaviour of an isotropic viscoelastic material will be specified by two independent functions of time. There are many ways of doing this. We will take either the creep or relaxation functions as the basic functions.

The constitutive equations proposed are :

$$T_{ij}(t) = \lambda \delta_{ij} \epsilon_{rr}(t) - \lambda_1 \delta_{ij} \int_{-\infty}^{t} \varphi_0(t-\tau) \frac{d\epsilon_{rr}(\tau)}{d\tau} d\tau$$

$$+ 2\mu \epsilon_{ij}(t) - 2\mu_1 \int_{-\infty}^{t} \varphi_1(t-\tau) \frac{d\epsilon_{ij}(\tau)}{d\tau} d\tau \quad \ldots 1.6.1$$

$$\epsilon_{ij}(t) = \frac{\lambda \delta_{ij}}{2\mu(3\lambda+2\mu)} T_{rr}(t) + \frac{\lambda_1 \delta_{ij}}{2\mu_1(3\lambda_1+2\mu_1)} \int_{-\infty}^{t} \psi_0(t-\tau) \frac{dT_{rr}(\tau)}{d\tau} d\tau$$

$$+ \frac{1}{2\mu} T_{ij}(t) + \frac{1}{2\mu_1} \int_{-\infty}^{t} \psi_1(t-\tau) \frac{dT_{ij}(\tau)}{d\tau} d\tau \quad \ldots 1.6.2$$

where δ_{ij} is the Kronecker delta, φ_0, φ_1 are relaxation functions and ψ_0, ψ_1 are creep functions.

Boltzmann (3) suggested that there is no elastic after-effect for purely dilational strains. This implies that the relaxation and creep functions are zero. Kolsky and Shi (4) proposed that this is probably a realistic assumption for a crystalline solid for which hydrostatic pressures merely change interatomic distances. Another assumption discussed by Kolsky and Shi is that $\varphi_h = \varphi_s$ and $\psi_h = \psi_s$, where φ_h, ψ_h are relaxation and creep functions under hydrostatic condition and φ_s, ψ_s are those under pure shear respectively. This hypothesis is clearly false for Newtonian fluids under shear.

2. FIELD EQUATIONS

Consider a body $\mathcal{B}_o$ bounded by the surface $\partial\mathcal{B}_o$ (as shown in Fig. 2.1). Let the body $\mathcal{B}_o$ be subjected to a system of forces, couples and displacements. Now imagine a portion $\mathcal{B}_1$ of $\mathcal{B}_o$ be separated from the rest of $\mathcal{B}_o$ by the surface $\partial\mathcal{B}_1$. Then, according to the generalised form of the stress principle of Euler and Cauchy, the effect of portion $\mathcal{B}_o-\mathcal{B}_1$ on $\mathcal{B}_1$ may be represented by a stress-vector $\underset{\sim}{t}(\underset{\sim}{n})$ and a couple-stress-vector $\underset{\sim}{m}(\underset{\sim}{n})$ acting over the surface $\partial\mathcal{B}_1$ bounding $\mathcal{B}_1$. Call any pair of distributions $(\underset{\sim}{t}, \underset{\sim}{m})$ and $(\underset{\sim}{t}_1, \underset{\sim}{m}_1)$ of force-stress and couple-stress equipollent if they yield the same values for the resultant force and moment on every part of the material body $\mathcal{B}_o$. Then, if the distribution $(\underset{\sim}{t}, \underset{\sim}{m})$ on surface $\partial\mathcal{B}_1$ be equipollent to the contact action (due to distributions $(\underset{\sim}{t}_1, \underset{\sim}{m}_1)$) of $(\mathcal{B}_o-\mathcal{B}_1)$ on $\mathcal{B}_1$, it is equivalent to saying that the part $\mathcal{B}_1$ cannot "feel" the difference when the portion $(\mathcal{B}_o-\mathcal{B}_1)$ is removed. Until now, most of the work in continuum mechanics has dealt with the non-polar case. The non-polar case is characterised by the conditions $\underset{\sim}{m}(\underset{\sim}{n}) = 0$ and $\underset{\sim}{l} = 0$ where $\underset{\sim}{l}$ is the body moment

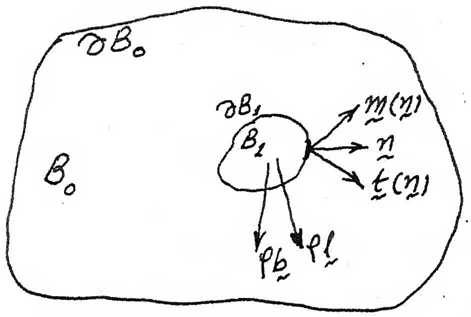


Fig. 2.1

per unit mass analogous to the body force per unit mass $\underset{\sim}{b}$. The basic principle of balance of moment of momentum with such a characterisation leads to the conclusion that the stress tensor T_{ij} obtained from $t_i = T_{ji} n_j$ is symmetric i.e. $T_{ji} = T_{ij}$. When couple-stresses and body moments are retained, the balance principle for angular momentum does not lead in general to the symmetry of stress tensor.

In fact there is no reason for neglecting the couple stress vector $\underset{\sim}{m}$. There may be cases where M_{ij} obtained from $m_i = M_{ji} n_j$ may have an insignificant magnitude or in certain classes of materials couple stress may not occur.

From Cauchy's first law (balance of linear momentum) one can see that the stress tensor T_{ij} exists for all materials. But from Cauchy's second law it can be seen that even if the body couple $\underset{\sim}{l}$ exists, the field equations can be satisfied by taking couple stresses to be zero. Thus the concept of body moment is essentially unrelated to the definition of couple-stress, just as the concept of body force is unrelated to the definition of force-stress.

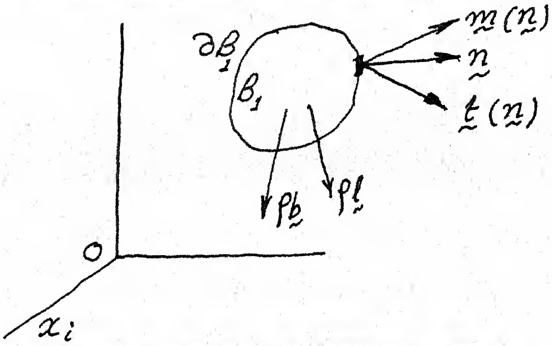
According to Truesdell and Toupin (5), body moments were first considered by Maxwell (1873), Larmor (1892), Combebiac (1902) and Duhem (1904).

Couple stresses have been considered by Voigt(1887), E. and F. Cosserat (1909), Heun (1913) and Gunther (1958).

Truesdell and Toupin have given a modern derivation of the Cosserat equations. Toupin (6) was the first to give a constitutive equation for couple stresses in elastic materials subject to finite deformations. Mindlin and Tiersten (7) linearised Toupin's constitutive equations and applied it to a number of problems such as effects of couple stresses on vibration, wave motion and stress concentration. They have also discussed the uniqueness of solutions. In a recent paper Stokes (8) has proposed a theory for couple stresses in fluids.

We have made an attempt here to extend the theory to viscoelastic materials.

2.1 Derivation of the Field equations



The equations for the conservation of mass, the Balance of momentum and moment of momentum and the Conservation of energy can be written as :

$$\frac{d}{dt}\int_{B_1} \rho \, dV = 0 \qquad \text{... 2.1.1}$$

$$\frac{d}{dt}\int_{B_1} \underset{\sim}{v} \rho \, dV = \int_{B_1} \underset{\sim}{b} \rho \, dV + \int_{\partial B_1} \underset{\sim}{t} \, ds \qquad \text{... 2.1.2}$$

$$\frac{d}{dt}\int_{B_1} \underset{\sim}{r} \times \underset{\sim}{v} \rho \, dv = \int_{B_1} (\underset{\sim}{r} \times \underset{\sim}{b} + \underset{\sim}{l}) \rho \, dv + \int_{\partial B_1} (\underset{\sim}{r} \times \underset{\sim}{t} + \underset{\sim}{m}) \, ds \qquad \ldots 2.1.3$$

$$\frac{d}{dt}\int_{B_1} (\tfrac{1}{2}\underset{\sim}{v}\cdot\underset{\sim}{v} + \varepsilon) \rho \, dv = \int_{B_1} (\underset{\sim}{b}\cdot\underset{\sim}{v} + \tfrac{1}{2}\underset{\sim}{l}\cdot\nabla\times\underset{\sim}{v} + q) \rho \, dv$$

$$+ \int_{\partial B_1} (\underset{\sim}{t}\cdot\underset{\sim}{v} + \tfrac{1}{2}\underset{\sim}{m}\cdot\nabla\times\underset{\sim}{v} - \underset{\sim}{h}\cdot\underset{\sim}{n}) \, ds \qquad \ldots 2.1.4$$

where $\frac{d}{dt}$ is the derivative, with respect to time, following a material particle. ρ is the mass density, $\underset{\sim}{r}$ is the spatial position vector from a fixed origin, $\underset{\sim}{v}$ is the material velocity $\frac{d\underset{\sim}{r}}{dt}$, ε is the energy per unit mass, $\underset{\sim}{h}$ is the influx of energy per unit area and q is the energy source dênsity per unit mass.

In each of the equations 2.1.1 to 2.1.4 the left hand term gives the quantity in balance, the surface integral in the right gives the influx relative to the material and the volume integral gives its source.

Consideration of the equilibrium of forces acting on an elementary tetrahedron, as the volume of the tetrahedron shrinks to zero, leads to the definition of the stress tensor T_{ij} as

$$t_i = T_{ji} n_j \qquad \ldots 2.1.5$$

An analogous treatment of the equilibrium of moments acting on the tetrahedron yields the definition of the

couple stress tensor M_{ij} as $m_i = M_{ji} n_j$ ... 2.1.6

Equation 2.1.1 on simplification (see Appendix 1) yields the continuity equation

$$\dot{\rho} + \rho \dot{u}_{i,i} \qquad \text{... 2.1.7}$$

Here $\dot{u}_i$ is the material velocity vector.

Using 2.1.5 and 2.1.7, equation 2.1.2 can be simplified to give the equation for balance of linear momentum (Cauchy's first law of motion, see Appendix 1) as :

$$T_{ji,j} + \rho b_i = \rho a_i \qquad \text{... 2.1.8}$$

where a_i is the material acceleration vector.

The equation for the balance of angular momentum (Cauchy's second law of motion) can now be written down, after simplifying equation 2.1.3 with the help of 2.1.5 to 2.1.8 (see Appendix 1), as :

$$e_{ijk} T_{jk} + M_{ji,j} + \rho l_i = 0 \qquad \text{... 2.1.9}$$

It can be seen from 2.1.8 that given an acceleration field and a body force distribution, the stress tensor has to exist. On the other hand equation 2.1.9 can be satisfied by taking couple stress tensor $M_{ji} = 0$ even if the body moment l_i is not zero.

The energy equation given in equation 2.1.4 on simplification by using 2.1.5 to 2.1.9 (Appendix 1) gives :

$$\rho\dot{\mathcal{E}} = T_{ji}^{S}\dot{\epsilon}_{ij} + M_{ij}^{D}\dot{K}_{ij} - h_{i,i} + \rho q \qquad \ldots 2.1.10$$

where T_{ji}^{S} is the symmetric part of the stress tensor.

M_{ij}^{D} is the deviatoric part of couple stress tensor given by $M_{ij}^{D} = \left(M_{ij} - \frac{1}{3}M_{rr}\delta_{ij}\right)$ ... 2.1.11

It is interesting to note that only the symmetric part of stress tensor and the deviatoric part of couple stress tensor appear in the energy equation.

Taking the curl of equation 2.1.9 we get

$$\nabla\times\left[e_{ijk}T_{jk} + M_{ji,j} + \rho l_i\right]$$
$$= e_{mni}e_{ijk}T_{jk,n} + e_{mni}M_{ji,jn} + e_{mni}(\rho l_i)_{,n} = 0$$

i.e. $$-(T_{mn} - T_{nm})_{,n} = e_{mni}M_{ji,jn} + e_{mni}(\rho l_i)_{,n}$$

$$T^{A}_{nm,n} = \frac{1}{2}\left\{e_{mni}M^{D}_{ji,jn} + e_{mni}(\rho l_i)_{,n}\right\}$$

... 2.1.12

Since $\nabla\times\nabla\cdot\underset{\approx}{M} = \nabla\times\nabla\cdot\underset{\approx}{M}^{D}$

Cauchy's first law of motion 2.1.8 becomes :

$$\rho a_i = T^S_{ji,j} + \frac{1}{2}\left[e_{ijk} M_{rk,rj} + e_{ijk}(\rho l_k)_{,j}\right] + \rho b_i \qquad \ldots 2.1.13$$

where $T^S_{ji} = \frac{1}{2}(T_{ji} + T_{ij})$

We can also get the antisymmetric part of the stress tensor from equation 2.1.9. Thus multiplying through by e_{mni} we get $e_{mni}\, e_{ijk} T_{jk} + e_{mni} M_{ji,j} + e_{mni}\rho l_i = 0$

i.e. $$T^A_{mn} = -\frac{1}{2}\left[e_{mni} M_{ji,j} + e_{mni}\rho l_i\right] \qquad \ldots 2.1.14$$

2.2 Kinematics of Deformation and flow:

If u_i be the components of displacement of a material particle, then for infinitesimal displacements, the components of strain are given by :

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad \ldots 2.2.1$$

The components of the strain tensor given by this small strain equation for $i \neq j$ are half the physical shear strain components.

The material rotation vector ω_i is defined as

$$\omega_i = \frac{1}{2} e_{ijk} u_{k,j} \qquad \ldots 2.2.2$$

The three components of ω_i give the small rotation of the material particle about the three axes. We define the curvature-twist tensor K_{ij} as the gradient of rotation vector ω_j

i.e. $$K_{ij} = \omega_{j,i} = \frac{1}{2} e_{jst} u_{t,si} \qquad \text{... 2.2.3}$$

Three diagonal components of K_{ij} give the twist of the material per unit length about the three axes. On the other hand, the off-diagonal components of K_{ij} are a measure of the curvatures induced in the various planes.

The rates of strain tensor, the vorticity and the curvature twist rate tensor are given respectively by :

$$\dot{\epsilon}_{ij} = \frac{1}{2}(\dot{u}_{i,j} + \dot{u}_{j,i}) \qquad \text{... 2.2.4}$$

$$\dot{\omega}_i = \frac{1}{2} e_{ijk} \dot{u}_{k,j} \qquad \text{... 2.2.5}$$

$$\dot{K}_{ij} = \dot{\omega}_{j,i} = \frac{1}{2} e_{jst} \dot{u}_{t,si} \qquad \text{... 2.2.6}$$

It follows from 2.2.3 and 2.2.6 that

$$K_{ii} = 0 \quad , \quad \dot{K}_{ii} = 0 \qquad \text{... 2.2.7}$$

Defining the antisymmetric part of the velocity gradient $\frac{1}{2}(\dot{u}_{i,j} - \dot{u}_{j,i}) = \dot{W}_{ij}$ as the vorticity tensor, it can be seen that :

$$\dot{W}_{ij} = e_{ijk} \dot{\omega}_k \qquad \text{... 2.2.8}$$

$$\dot{\omega}_k = \frac{1}{2} e_{ijk} \dot{W}_{ij} \qquad \text{... 2.2.9}$$

3. CONSTITUTIVE EQUATIONS

3.1. The energy equation came out to be

$$\rho \dot{\mathcal{E}} = T_{ij}^{s} \dot{\epsilon}_{ij} + M_{ij}^{D} \dot{\kappa}_{ij} - h_{i,i} + \rho q \qquad 2.1.10$$

The antisymmetric part of the stress tensor and the trace of $\underset{\approx}{M}$ do not contribute to the change in $\dot{\mathcal{E}}$. The term $T_{ij}^{s} \dot{\epsilon}_{ij}$ remains the same as in non-polar case. The term $M_{ij}^{D} \dot{\kappa}_{ij}$ has appeared as an additional term and shows that only deviatoric part of the couple stress tensor contributes to $\dot{\mathcal{E}}$. It is also interesting to note that the force stress and couple stress terms appear separately in the energy equation.

Mindlin and Tiersten (7) have given linear constitutive equations for the polar case in elasticity. Stokes (8) has given the same for fluids. We will use these results to propose constitutive equations for the viscoelastic case.

Considering the constitutive equations proposed by Mindlin and Tiersten and Stokes we propose the following constitutive equations for the symmetric part of stress tensor and deviatoric part of couple stress tensor for isotropic viscoelastic materials :-

$$T_{ij}^{s} = \lambda \epsilon_{rr} \delta_{ij} + 2\mu \epsilon_{ij} - \lambda_1 \delta_{ij} \int_{-\infty}^{t} \varphi_0(t-\tau)\, \dot{\epsilon}_{rr}(\tau)\, d\tau$$

$$- 2\mu_1 \int_{-\infty}^{t} \varphi_1(t-\tau)\, \dot{\epsilon}_{ij}(\tau)\, d\tau \qquad \ldots 3.1.1$$

$$M_{ij}^{D} = 4\eta K_{ij} + 4\eta' K_{ji} - 4\eta_1 \int_{-\infty}^{t} \varphi_2(t-\tau)\dot{K}_{ij}(\tau)\,d\tau - 4\eta_1' \int_{-\infty}^{t} \varphi_3(t-\tau)\dot{K}_{ji}(\tau) \quad \ldots 3.1.2$$

The functions $\varphi_0, \varphi_1, \varphi_2, \varphi_3$ are relaxation functions with forms similar to those given in Section 1.

The constants λ, μ, λ_1, μ_1, η, η', η_1, η_1', may be constants or in general functions of h_i, q etc.

$\lambda, \mu, \lambda_1, \mu_1$ have dimensions of Force/Area i.e. $\left(\frac{M}{LT^2}\right)$

$\eta, \eta', \eta_1, \eta_1'$ have dimensions of Force i.e. $\left(\frac{ML}{T^2}\right)$

The ratios between the second set and first set of constants have the dimensions of L^2.

The antisymmetric part of stress tensor and the trace of couple stress tensor are left undetermined by the constitutive equations and can be determined from the boundary conditions. We have already seen that these quantities do not come in in the energy equation.

2.2 Summary of all the equations

So far we have come across the following equations :

$\dot{\rho} + \rho \dot{u}_{i,i} = 0$ one equation 2.1.7

$T_{ji,j} + \rho b_i = \rho a_i$ Three equations 2.1.8

$e_{ijk}\,T_{jk} + M_{ji,j} + \rho l_i = 0$ Three equations ... 2.1.9

$\rho\dot{\mathcal{E}} = T_{ji}^{S}\dot{\epsilon}_{ij} + M_{ij}^{D}\dot{K}_{ij} - h_{i,i} + \rho q$ One equation ... 2.1.10

$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ Six independent eqns. 2.2.1

$\dot{\epsilon}_{ij} = \frac{1}{2}(\dot{u}_{i,j} + \dot{u}_{j,i})$ Six independent eqns. 2.2.4

$K_{ij} = \omega_{j,i} = \frac{1}{2} e_{jst} u_{t,si}$ Nine equations ... 2.2.3

$\dot{K}_{ij} = \dot{\omega}_{j,i} = \frac{1}{2} e_{jst} \dot{u}_{t,si}$ Nine equations ... 2.2.6

$$T_{ij}^{S} = \lambda\epsilon_{rr}\delta_{ij} + 2\mu\epsilon_{ij} - \lambda_1\delta_{ij}\int_{-\infty}^{t}\varphi_0(t-\tau)\,\dot{\epsilon}_{rr}(\tau)\,d\tau - 2\mu_1\int_{-\infty}^{t}\varphi_1(t-\tau)\,\dot{\epsilon}_{ij}(\tau)\,d\tau$$

Six independent eqns. 3.1.1

$$M_{ij}^{D} = 4\eta K_{ij} + 4\eta' K_{ji} - 4\eta_1\int_{-\infty}^{t}\varphi_2(t-\tau)\,\dot{K}_{ij}(\tau)\,d\tau - 4\eta_1'\int_{-\infty}^{t}\varphi_3(t-\tau)\,\dot{K}_{ji}(\tau)\,d\tau$$

Eight independent eqns. 3.1.2

$K_{ii} = 0\ ,\quad \dot{K}_{ii} = 0$ Two equations ... 2.2.7

We add here the following supplementary equations :

$\nabla\times\nabla\underset{\sim}{\omega} = e_{ijk}\,\omega_{r,kj} = e_{ijk}\,K_{kr,j} = 0$ ... 3.2.1

$\nabla\times\nabla\underset{\sim}{\dot{\omega}} = e_{ijk}\,\dot{\omega}_{r,kj} = e_{ijk}\,\dot{K}_{kr,j} = 0$ ... 3.2.2

and $$\nabla \times \underset{\approx}{\epsilon} \times \nabla = e_{ijk} e_{rst} \epsilon_{sk,jt} = 0 \qquad \ldots 3.2.3$$

$$\nabla \times \underset{\approx}{\dot{\epsilon}} \times \nabla = e_{ijk} e_{rst} \dot{\epsilon}_{sk,jt} = 0 \qquad \ldots 3.2.4$$

Nine relations under each of 3.2.1 and 3.2.2 give the compatibility conditions for K_{ij}, $\dot{K}_{ij}$, whereas six independe relations under each of 3.2.3 and 3.2.4 give the compatibility conditions for ϵ_{ij}, $\dot{\epsilon}_{ij}$ respectively.

3.3 Equation of motion in terms of displacement components

From Cauchy's second law of motion we get

$$T^{A}_{ji,j} = \frac{1}{2}\left[e_{ijk} M_{rk,rj} + e_{ijk}(\rho l_k)_{,j} \right] \qquad \ldots 2.1.12$$

Substituting in Cauchy's first law of motion :

$$\rho a_i = T^{S}_{ji,j} + \frac{1}{2}\left[e_{ijk} M_{rk,rj} + e_{ijk}(\rho l_k)_{,j} \right] + \rho b_i \qquad \ldots 2.1.13$$

Making use of the constitutive equations for T^{S}_{ji} and M^{D}_{rk} the above equation is written down as :

$$\rho a_i = \left\{ \lambda \epsilon_{rr} \delta_{ij} + 2\mu \epsilon_{ij} - \lambda_1 \delta_{ji} \int_{-\infty}^{t} \varphi_0(t-\tau) \dot{\epsilon}_{rr}(\tau) d\tau - 2\mu_1 \int_{-\infty}^{t} \varphi_1(t-\tau) \dot{\epsilon}_{ij}(\tau) d\tau \right\}_{,j}$$
$$+ \frac{1}{2} e_{ijk} \left\{ 4\eta K_{rk} + 4\eta' K_{kr} - 4\eta_1 \int_{-\infty}^{t} \varphi_2(t-\tau) \dot{K}_{rk}(\tau) d\tau - 4\eta'_1 \int_{-\infty}^{t} \varphi_3(t-\tau) \dot{K}_{kr}(\tau) d\tau \right\}_{,rj}$$
$$+ \frac{1}{2} e_{ijk} (\rho l_k)_{,j} + \rho b_i$$

To simplify the above equation we proceed (term by term) as follows with the assumption that material is homogeneous and incompressible.

(i) $(\lambda \epsilon_{rr})_{,j} \delta_{ji} = \lambda \epsilon_{rr,i} = \lambda u_{r,ri}$

(ii) $(2\mu \epsilon_{ji})_{,j} = \mu (u_{j,ij} + u_{i,jj})$

(iii) $\left\{-\lambda_1 \delta_{ji} \int_{-\infty}^{t} \varphi_0(t-\tau) \dot{\epsilon}_{rr}(\tau)\, d\tau\right\}_{,j} = -\lambda_1 \int_{-\infty}^{t} \varphi_0(t-\tau) \dot{u}_{r,ri}(\tau)\, d\tau$

(iv) $\left\{-2\mu_1 \int_{-\infty}^{t} \varphi_1(t-\tau) \dot{\epsilon}_{ij}(\tau)\, d\tau\right\}_{,j} = -\mu_1 \int_{-\infty}^{t} \varphi_1(t-\tau) (\dot{u}_{i,j} + \dot{u}_{j,i})_{,j}\, d\tau$

$$= -\mu_1 \int_{-\infty}^{t} \varphi_1(t-\tau) (\dot{u}_{i,jj} + \dot{u}_{j,ij})\, d\tau$$

(v) $\frac{1}{2} e_{ijk} 4\eta (K_{rk})_{,rj} = 2\eta e_{ijk} \frac{1}{2} e_{kst}\, u_{t,srrj}$

$$= \eta (u_{j,irrj} - u_{i,jrrj})$$

(vi) $\frac{1}{2} e_{ijk} (4\eta' K_{kr})_{,rj} = 2\eta' e_{ijk} \frac{1}{2} e_{rst}\, u_{t,skrj} = 0$

(vii) $-\frac{1}{2} e_{ijk} \left\{4\eta_1 \int_{-\infty}^{t} \varphi_2(t-\tau) \dot{K}_{rk}(\tau)\, d\tau\right\}_{,rj}$

$$= -2\eta_1 e_{ijk} \int_{-\infty}^{t} \varphi_2(t-\tau) \frac{1}{2} e_{kst}\, \dot{u}_{t,srrj}\, d\tau$$

$$= \eta_1 \int_{-\infty}^{t} \varphi_2(t-\tau) (\dot{u}_{i,rrjj} - \dot{u}_{j,jirr})\, d\tau$$

(viii) $$-\frac{1}{2} e_{ijk}\left\{4\eta_1'\int_{-\infty}^{t}\varphi_3(t-\tau)\dot{\kappa}_{kr}(\tau)\,d\tau\right\}_{,rj}$$

$$= -2\eta_1'\int_{-\infty}^{t}\varphi_3(t-\tau)\, e_{ijk}\, e_{rst}\, \dot{u}^{(\tau)}_{t,skrj}\, d\tau = 0$$

The equation of motion (Eqn. 3.3.1), after substitution of simplified terms from (i) to (viii) above, comes out as :

$$\rho a_i = \lambda u_{r,ri} + \mu\left(u_{j,ji} + u_{i,jj}\right) - \lambda_1\int_{-\infty}^{t}\varphi_0(t-\tau)\dot{u}^{(\tau)}_{r,ri}\, d\tau$$
$$-\mu_1\int_{-\infty}^{t}\varphi_1(t-\tau)\left(\dot{u}_{i,jj} + \dot{u}_{j,ji}\right) d\tau + \eta\left(u_{j,jrri} - u_{i,jjrr}\right)$$
$$+\eta_1\int_{-\infty}^{t}\varphi_2(t-\tau)\left(\dot{u}_{i,rrjj} - \dot{u}_{j,jirr}\right) d\tau + \frac{1}{2} e_{ijk}(\rho l_k)_{,j} + \rho b_i$$

This can be written down in more compact form as :

$$\rho a_i = (\lambda+\mu)\, u_{r,ri} + \mu u_{i,rr} + \eta\left(u_{j,jrri} - u_{i,jjrr}\right)$$
$$-\int_{-\infty}^{t}\left\{\lambda_1\varphi_0(t-\tau) + \mu_1\varphi_1(t-\tau)\right\}\dot{u}_{r,ri} - \mu_1\int_{-\infty}^{t}\varphi_1(t-\tau)\,\dot{u}_{i,jj}\, d\tau$$
$$+\eta_1\int_{-\infty}^{t}\varphi_2(t-\tau)\left(\dot{u}_{i,rrjj} - \dot{u}_{j,jirr}\right) d\tau + \frac{1}{2} e_{ijk}\,(\rho l_k)_{,j} + \rho b_i$$

or in the vector notation as :

$$\rho \underset{\sim}{a} = (\lambda+\mu+\eta\nabla^2)\nabla\nabla\cdot\underset{\sim}{u} + \mu\nabla^2\underset{\sim}{u} - \eta\nabla^4\underset{\sim}{u}$$

$$-\mu_1\int_{-\infty}^{t}\varphi_1(t-\tau)\nabla^2\dot{\underset{\sim}{u}}(\tau)d\tau + \eta_1\int_{-\infty}^{t}\varphi_2(t-\tau)\nabla^4\dot{\underset{\sim}{u}}(\tau)d\tau$$

$$-\int_{-\infty}^{t}\left\{\lambda_1\varphi_0(t-\tau)+\mu_1\varphi_1(t-\tau)+\eta_1\varphi_2(t-\tau)\nabla^2\right\}\nabla\nabla\cdot\dot{\underset{\sim}{u}}(\tau)d\tau$$

$$+\frac{1}{2}\nabla\times(\rho\underset{\sim}{l}) + \rho\underset{\sim}{b} \qquad \cdots 3.3.3$$

3.4 Antisymmetric Part of Stress Tensor :

To evaluate the antisymmetric part of the stress tensor in terms of the displacement and velocity field, we start from Cauchy's second law of motion and get :

$$T^A_{mn} = -\frac{1}{2}\left[e_{mni}M_{ji,j} + e_{mni}\rho l_i\right] \qquad \cdots 2.1.14$$

Substituting the constitutive equation for M_{ji} we get

$$T^A_{mn} = -\frac{1}{2}e_{mni}\left[4\eta K_{ji} + 4\eta' K_{ij} - 4\eta_1\int_{-\infty}^{t}\varphi_2(t-\tau)\dot{K}_{ji}(\tau)d\tau - 4\eta_1'\int_{-\infty}^{t}\varphi_3(t-\tau)\dot{K}_{ij}d\tau\right]_{,j}$$

$$-\frac{1}{2}e_{mni}\rho l_i \qquad \cdots 3.4.1$$

Simpligying (term by term) we get :

(i) $-\frac{1}{2} e_{mni} (4\eta K_{ji})_{,j} = -\eta\, e_{mni} e_{irs} u_{s,rjj}$

$$= -\eta (u_{n,m} - u_{m,n})_{,jj} = 2\eta W_{mn,jj}$$

where $W_{mn} = \frac{1}{2}(u_{m,n} - u_{n,m})$ is the antisymmetric part of the displacement gradient.

(ii) $-\frac{1}{2} e_{mni} (4\eta' K_{ij})_{,j} = -\eta' e_{mni} e_{jrs} u_{s,rij} = 0$

(iii) $\frac{1}{2} e_{mni} \left\{ 4\eta_1 \int_{-\infty}^{t} \varphi_2(t-\tau)\, \dot{K}_{ji}(\tau)\, d\tau \right\}_{,j}$

$$= \eta_1 \int_{-\infty}^{t} \varphi_2(t-\tau)\, e_{mni} e_{irs}\, \dot{u}_{s,rjj} = -2\eta_1 \int_{-\infty}^{t} \varphi_2(t-\tau)\, \dot{W}_{mn,jj}(\tau)\, d\tau$$

(iv) $\frac{1}{2} e_{mni} \left\{ 4\eta_1' \int_{-\infty}^{t} \varphi_3(t-\tau)\, \dot{K}_{ij}\, d\tau \right\}_{,j} = 0$ follows from step under (ii) above.

After substitution in 3.4.1, the expression for T_{mn}^{A} becomes :

$$T_{mn}^{A} = 2\eta W_{mn,jj} - 2\eta_1 \int_{-\infty}^{t} \varphi_2(t-\tau)\, \dot{W}_{mn,jj}(\tau)\, d\tau - \frac{1}{2} e_{mni}\, \rho l_i \qquad \ldots 3.4.2$$

4. Method of Solution :

4.1 Consider the equation of motion (Eq. 3.3.3)

$$\rho \underset{\sim}{a}(t) = (\lambda+\mu+\eta\nabla^2)\nabla\nabla.\underset{\sim}{u}(t) + \mu\nabla^2\underset{\sim}{u}(t) - \eta\nabla^4\underset{\sim}{u}(t)$$
$$-\mu_1\int_{-\infty}^{t}\varphi_1(t-\tau)\nabla^2\dot{\underset{\sim}{u}}(\tau)\,d\tau + \eta_1\int_{-\infty}^{t}\varphi_2(t-\tau)\nabla^4\dot{\underset{\sim}{u}}(\tau)\,d\tau$$
$$-\int_{-\infty}^{t}\{\lambda_1\varphi_0(t-\tau)+\mu_1\varphi_1(t-\tau)+\eta_1\varphi_2(t-\tau)\nabla^2\}\nabla\nabla.\dot{\underset{\sim}{u}}(\tau)\,d\tau \quad \ldots 4.1.1$$

The body forces and body moments have been neglected in the above equation.

Taking Laplace transformation of Eqn. 4.1.1 we get

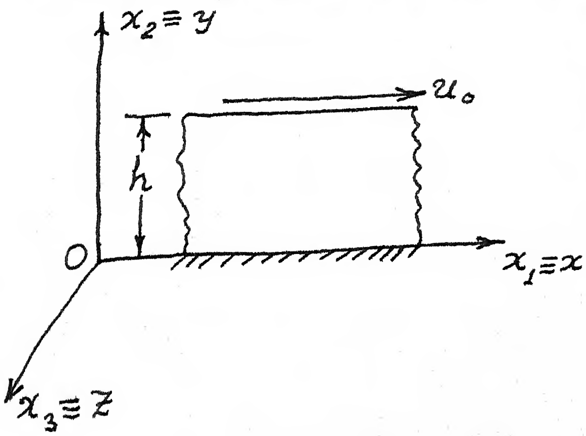
$$\rho\underset{\sim}{A}(s) = (\lambda+\mu+\eta\nabla^2)\nabla\nabla.\underset{\sim}{U}(s) + \mu\nabla^2\underset{\sim}{U}(s) - \eta\nabla^4\underset{\sim}{U}(s)$$
$$-\mu_1\bar{\Phi}_1(s)\nabla^2\dot{\underset{\sim}{U}}(s) + \eta_1\bar{\Phi}_2(s)\nabla^4\dot{U}(s)$$
$$-\left[\lambda_1\bar{\Phi}_0(s)+\mu_1\bar{\Phi}_1(s)+\eta_1\bar{\Phi}_2(s)\nabla^2\right]\nabla\nabla.\dot{\underset{\sim}{U}}(s) \quad \ldots 4.1.2$$

The terms A, $\underset{\sim}{U}$, $\dot{\underset{\sim}{U}}$, $\bar{\Phi}_i$ are the Laplace transforms of the respective quantities.

The boundary conditions may be transformed in the same way. With the help of such a transformation the time variable is removed from the equation, which can be solved and the subsequent inversion will give the time dependent solution for the viscoelastic problem.

4.2 Particular Case:

Let us consider a problem of an infinite plate of thickness h fixed at its bottom face. At time t = 0 let the top face be suddenly displaced in the x_1 direction (in a rect. Cartesian x_i frame) by a small amount u_o and then its position be maintained. We are faced with the problem of finding out how the displacement field varies with time inside the plate.



Field equation satisfied by $\underset{\sim}{u}$ is given by eqn. 4.1.1 and boundary conditions are as follows:

$$u_1(0,t) = 0$$

$$\dot{u}_1(0,t) = 0$$

$$u_1(y,0) = 0, \quad 0 \leq y < h$$

$$\dot{u}_1(y,0) = 0, \quad 0 \leq y < h$$

$$u_1(h,t) = u_o H(t)$$

H(o) is the Heaviside function prescribed at t = 0.

We note that $\underset{\sim}{u} = \hat{i}\, u_1(y,t)$, $\hat{i}$ being an unit vector in the direction of x_1.

Then $\nabla \cdot \underset{\sim}{u} = u_{1,1} = 0$ and similarly $\nabla \cdot \dot{\underset{\sim}{u}} = 0$

$$\nabla^2 \underset{\sim}{u} = \hat{i}\, u_{1,22} = \hat{i} \frac{\partial^2 u_1}{\partial y^2} \quad ; \quad \nabla^4 \underset{\sim}{u} = \hat{i}\, u_{1,2222} = \hat{i} \frac{\partial^4 u_1}{\partial y^4}$$

With this equation of motion (4.1.1) reduces to :

$$\rho a_1 = \mu \frac{\partial^2 u_1}{\partial y^2} - \eta \frac{\partial^4 u_1}{\partial y^4} - \mu_1 \int_{-\infty}^{t} \varphi_1(t-\tau) \frac{\partial^2 \dot{u}_1}{\partial y^2} d\tau + \eta_1 \int_{-\infty}^{t} \varphi_2(t-\tau) \frac{\partial^4 \dot{u}_1}{\partial y^4} d\tau$$

$$\rho a_2 = 0 \quad ; \quad \rho a_3 = 0 \qquad \ldots 4.2.1$$

Taking Laplace transform of eqn. 4.2.1 we get

$$\rho A_1 = \mu \frac{\partial^2 U_1}{\partial y^2} - \eta \frac{\partial^4 U_1}{\partial y^4} - \mu_1 \bar{\Phi}_1 \frac{\partial^2 \dot{U}_1}{\partial y^2} + \eta_1 \bar{\Phi}_2 \frac{\partial^4 \dot{U}_1}{\partial y^4} \qquad \ldots 4.2.2$$

$$A_1 = \mathcal{L}\, a_1 = \mathcal{L} \left\{ \frac{\partial \dot{u}_1}{\partial t} + \dot{u}_1 \frac{\partial \dot{u}_1}{\partial x_1} + \dot{u}_2 \frac{\partial \dot{u}_1}{\partial x_2} + \dot{u}_3 \frac{\partial \dot{u}_1}{\partial x_3} \right\} = \mathcal{L} \frac{\partial \dot{u}_1}{\partial t}$$

$$\equiv \mathcal{L} \frac{\partial^2 u_1}{\partial t^2} = S^2 U_1 - S\, u_1(0) - \dot{u}_1(0) \qquad \ldots 4.2.3$$

$$\dot{U}_1 = \mathcal{L}\, \dot{u}_1 \equiv \mathcal{L} \frac{\partial u_1}{\partial t} = SU - u_1(0) \qquad \ldots 4.2.4$$

Taking Laplace transforms of the boundary conditions we get

$$\mathcal{L}\, u_1(0,t) = U_1(0,S) = 0$$

$$\mathcal{L}\, \dot{u}_1(0,t) = \dot{U}_1(0,S) = 0$$

$$\mathcal{L}\, u_1(h,t) = \mathcal{L}\, u_0 H(t) = \frac{u_0}{S} \qquad \ldots 4.2.5$$

$$\text{i,e} \quad U_1(h,S) = \frac{u_0}{S}$$

Using the initial conditions $u_1(y,0) = 0$, $\dot{u}_1(y,0) = 0$ we get from 4.2.3 and 4.2.4

$$A_1 = S^2 U_1$$

$$\dot{U} = S U_1$$

Substituting in equation 4.2.2,

$$\rho s^2 U_1 = \mu \frac{\partial^2 U_1}{\partial y^2} - \eta \frac{\partial^4 U_1}{\partial y^4} - s\mu_1 \bar{\varphi}_1(s) \frac{\partial^2 U_1}{\partial y^2} + s\eta_1 \bar{\varphi}_2(s) \frac{\partial^4 U_1}{\partial y^4} \quad \ldots 4.2.6$$

Further simplification can be obtained when the relaxation functions φ_1, φ_2 are known.

Let us take the case when $\varphi_1(t) = 1 - e^{-\alpha_1 t}$

$$\bar{\varphi}_1(s) = \mathcal{L}\left(1 - e^{-\alpha_1 t}\right) = \frac{1}{s} - \frac{1}{s+\alpha_1} = \frac{\alpha_1}{s(s+\alpha_1)}$$

Similarly taking $\varphi_2(t) = 1 - e^{-\alpha_2 t}$ we get $\bar{\varphi}_2(s) = \dfrac{\alpha_2}{s(s+\alpha_2)}$

Substituting in eqn. 4.2.6 for $\bar{\varphi}_1$ and $\bar{\varphi}_2$

$$\rho s^2 U_1 = \left(\mu - \mu_1 \frac{\alpha_1}{s+\alpha_1}\right) \frac{\partial^2 U_1}{\partial y^2} - \left(\eta - \eta_1 \frac{\alpha_2}{s+\alpha_2}\right) \frac{\partial^4 U_1}{\partial y^4} \quad \ldots 4.2.7$$

or

$$\rho s^2 U_1 = a \frac{\partial^2 U_1}{\partial y^2} - b \frac{\partial^4 U_1}{\partial y^4} \quad \ldots 4.2.8$$

where meaning of a and b are obvious.

Equation 4.2.8 has got a solution of the form :

$$U_1 = C_1 e^{m_1 y} + C_2 e^{m_2 y} + C_3 e^{m_3 y} + C_4 e^{m_4 y} \quad \ldots 4.2.9$$

where m_i are given by $m_i = \pm \sqrt{\frac{1}{2}\left\{\frac{a}{b} \pm \sqrt{\left(\frac{a}{b}\right)^2 - 4\frac{\rho s^2}{b}}\right\}}$

which rewritten in terms of the usual quantities, replacing a and b come out as :

$$m_i = \pm \left\{ \frac{1}{2} \; \frac{\mu - \dfrac{\mu_1 \alpha_1}{s+\alpha_1}}{\eta - \dfrac{\eta_1 \alpha_2}{s+\alpha_2}} \right\}^{1/2} \left\{ 1 \pm \sqrt{1 - 4\rho s^2 \left(\frac{\eta - \dfrac{\eta_1 \alpha_2}{s+\alpha_2}}{\left\{\mu - \dfrac{\mu_1 \alpha_1}{s+\alpha_1}\right\}^2} \right)} \right\}^{1/2}$$

Four boundary conditions will be needed to solve for the four constants of the solution 4.2.9. Two of the boundary conditions are :

$$U_1(0,s) = 0$$

$$U_1(h,s) = \frac{u_0}{s}$$

Other two may be taken from the physical conditions that couple stresses are zero on the planes y = 0 and y = h requiring that K_{2i} (0,s) = 0 and $K_{2i}(h,s) = 0$

or $U_{1,22}(0,s) = 0$ and $U_{1,22}(h,s) = 0$

DISCUSSION

The constitutive equations proposed for visco-elastic materials in section 1, have been chosen after considering the equations for an elastic solid and a viscous fluid. It has been assumed that the stress at the present time can be expressed as the sum of a contribution due to the strain at the present time and an integrated effect due to the history of strain rate. The relations are assumed to be linear and it is also assumed that the integrated effect due to history is only due to the first time derivative of strain. These equations are expected to approximate material behaviour in cases where the displacement gradient is small and also both its space and time derivatives are small.

The equations of motion in terms of displacement components are of the fourth order. The validity of the superposition principle used in writing down the three dimensional constitutive equation starting from the one-dimensional postulate is the same as for non-polar visco-elasticity.

The material constants can be determined experimentally only after suitable boundary value problems have been solved. A scheme for solving boundary value problems using the Laplace Transform has been indicated. However, it is not easy to invert the transforms which come up in this method.

APPENDIX 1

Balance Principle of Mass :

$$\frac{d}{dt}\int_B \rho \, dV = 0 \qquad \text{... 2.1-1}$$

But $\displaystyle \frac{d}{dt}\int_B \rho \, dV = \int_B \rho_{,t} \, dV + \int_{\partial B} \rho \, \dot{\underset{\sim}{r}} \cdot \underset{\sim}{n} \, ds$

$(\cdots)_{,t}$ indicates partial derivative with respect to t and dot as superscript shows the material time derivative. Noting $\dot{\underset{\sim}{r}} \equiv \underset{\sim}{v} \equiv \dot{u}_i$ in a rectangular cartesian frame x_i we get from above

$$\frac{d}{dt}\int_B \rho \, dV = \int_B \rho_{,t} \, dV + \int_{\partial B} \rho \, \dot{u}_i n_i \, ds$$

$$= \int_B \rho_{,t} \, dV + \int_B (\rho \dot{u}_i)_{,i} \, dV \qquad \text{(using divergence theorem)}$$

$$= \int_B \left[\rho_{,t} + (\rho \dot{u}_i)_{,i} \right] dV \qquad \text{from 2.1-1}$$

The above relation being true for any arbitrary chunk of material body B where all the fields are continuously differentiable, gives us the equivalent result

$$\rho_{,t} + (\rho \dot{u}_i)_{,i} = 0$$

or $\quad \left(\rho_{,t} + \dot{u}_i \rho_{,i} \right) + \rho \dot{u}_{i,i} = 0$

or $\quad \dot{\rho} + \rho \dot{u}_{i,i} = 0 \quad$ or $\quad \dot{\rho} + \rho \nabla \cdot \dot{\underset{\sim}{u}} = 0$

The balance principle for mass, expressing the conservation of mass, is given by the continuity equation derived above.

<u>Balance Principle of Linear Momentum (Cauchy's 1st law of motion)</u> :

Recall the equation

$$\frac{d}{dt}\int_{B} \underset{\sim}{v}\, \rho\, dV = \int_{\partial B} \underset{\sim}{t}\, ds + \int_{B} \underset{\sim}{b}\, \rho\, dV \qquad \text{... 2.1-2}$$

Left hand side $= \int_{B} (\rho\, dV) \frac{d}{dt}\underset{\sim}{v}$ (using continuity eqn.)

$$= \int_{B} \rho \ddot{u}_i\, dV = \int_{B} \rho a_i\, dV$$

where $\ddot{u}_i = a_i$ the acceleration of a material particle.

Right hand side $= \int_{\partial B} T_{ji} n_j\, ds + \int_{B} \rho b_i\, dV$

$= \int_{B} T_{ji,j}\, dV + \int_{B} \rho b_i\, dV$ (using divergence theorem)

Thus 2.1-2 becomes

$$\int_{B} \left(T_{ji,j} + \rho b_i - \rho a_i\right) dV = 0$$

This relation being true for arbitrary body B the integrand must vanish

or $\quad T_{ji,j} + \rho b_i = \rho a_i$

In vector notation $\quad \nabla \cdot \underset{\approx}{T} + \rho \underset{\sim}{b} = \rho \underset{\sim}{a}$

Balance Principle of Angular Momentum (Cauchy's 2nd Law):

Recall the eqn.

$$\frac{d}{dt}\int_B \underset{\sim}{r} \times \underset{\sim}{v}\, \rho dV = \int_{\partial B}(\underset{\sim}{r}\times\underset{\sim}{t}+\underset{\sim}{m})ds + \int_B(\underset{\sim}{r}\times\underset{\sim}{b}+\underset{\sim}{l})\rho dV \quad \ldots 2.1\text{-}3$$

Left hand side $= \int_B(\rho dV)\frac{d}{dt}(\underset{\sim}{r}\times\underset{\sim}{v})$ Using continuity eqn.

$$= \int_B \rho dV\,(\underset{\sim}{v}\times\underset{\sim}{v} + \underset{\sim}{r}\times\dot{\underset{\sim}{v}}) = \int_B \underset{\sim}{r}\times\dot{\underset{\sim}{v}}\cdot\rho dV$$

$= \int_B e_{ijk}\, x_j\, a_k\, \rho dV$, e_{ijk} is the permutation symbol in cartesian tensor.

The surface integral in the right hand side of 2.1-3

$$= \int_{\partial B}(\underset{\sim}{r}\times\underset{\sim}{t}+\underset{\sim}{m})\, ds = \int_{\partial B}\left[\underset{\sim}{r}\times(\underset{\sim}{n}\cdot\underset{\approx}{T}) + \underset{\sim}{n}\cdot\underset{\approx}{M}\right]ds$$

$$= \int_{\partial B}\left(-\underset{\sim}{n}\cdot\underset{\approx}{T}\times\underset{\sim}{r} + \underset{\sim}{n}\cdot\underset{\approx}{M}\right)ds$$

$$= -\int_{\partial B}\left[e_{ijk}\, T_{rj}\, n_r\, x_k - M_{ri}\, n_r\right]ds$$

$= -\int_B (e_{ijk}\, T_{rj}\, x_k - M_{ri})_{,r}\, dV$ Using divergence theorem.

$$= -\int_B\left(e_{ijk}\, x_k\, T_{rj,r} + e_{ijk}\, T_{rj}\,\delta_{kr} - M_{ri,r}\right)dV$$

The volume integral in the right hand side of 2.1-3

$$= \int_B (\underset{\sim}{r} \times \underset{\sim}{b} + \underset{\sim}{l}) \rho \, dV = \int_B (e_{ijk} x_j b_k + l_i) \rho \, dV$$

The complete eqn. 2.1-3 becomes :

$$\int_B e_{ijk} x_j (\rho a_k - T_{rk,r} - \rho b_k) \, dV = \int_B (-e_{ijk} T_{kj} + M_{ri,r} + \rho l_i) \, dV$$

Left hand side vanish by virtue of balance of linear momentum. The right side yields

$$e_{ijk} T_{jk} + M_{ji,j} + \rho l_i = 0$$

or $\underset{\sim}{T}_A + \nabla \cdot \underset{\approx}{M} + \rho \underset{\sim}{l} = 0$ where $\underset{\sim}{T}_A$ is the axial vector of $\underset{\approx}{T}$

<u>Balance Principle of Energy</u> :

Recall the eqn. :

$$\frac{d}{dt} \int_B \left(\tfrac{1}{2} \underset{\sim}{v} \cdot \underset{\sim}{v} + \varepsilon \right) \rho \, dV = \int_{\partial B} \left(\underset{\sim}{t} \cdot \underset{\sim}{v} + \tfrac{1}{2} \underset{\sim}{m} \cdot \nabla \times \underset{\sim}{v} - \underset{\sim}{h} \cdot \underset{\sim}{n} \right) ds$$

$$+ \int_B \left(\underset{\sim}{b} \cdot \underset{\sim}{v} + \tfrac{1}{2} \underset{\sim}{l} \cdot \nabla \times \underset{\sim}{v} + q \right) \rho \, dV \quad \ldots 2.1\text{-}4$$

Left side $= \int_B (\rho \, dV) \frac{d}{dt} \left(\tfrac{1}{2} \underset{\sim}{v} \cdot \underset{\sim}{v} + \varepsilon \right)$ Using continuity eqn.

$$= \int_B \left(\underset{\sim}{v} \cdot \dot{\underset{\sim}{v}} + \dot{\varepsilon} \right) \rho \, dV = \int_B \left(\dot{u}_i a_i + \dot{\varepsilon} \right) \rho \, dV$$

Note that $\int_{\partial B} \tfrac{1}{2} \underset{\sim}{m} \cdot \nabla \times \underset{\sim}{v} \, ds = \int_{\partial B} \tfrac{1}{2} (M_{ji} n_j)(e_{ist} \dot{u}_{t,s}) \, ds$

$$= \frac{1}{2}\int_B (e_{ist}\,\mathring{u}_{t,s}\,M_{ji})_{,j}\,dV$$ (using divergence theorem)

$$= \frac{1}{2}\int_B (e_{ist}\,\mathring{u}_{t,sj}\,M_{ji} + e_{ist}\,\mathring{u}_{t,s}\,M_{ji,j})\,dV$$

$$= \int_B (\dot{\omega}_{i,j}\,M_{ji} + \dot{\omega}_i\,M_{ji,j})\,dV$$

where $\omega_i = \frac{1}{2}\nabla\times\mathring{\underset{\sim}{u}}$ is the vorticity vector.

Similarly we may get $\int_V (\frac{1}{2}\underset{\sim}{l}\cdot\nabla\times\underset{\sim}{v})\,\rho\,dV = \int_V \dot{\omega}_i\,l_i\,\rho\,dV$

Substituting in eqn. 2.1-4 we get

$$\int_B (\mathring{u}_i a_i + \dot{\varepsilon})\rho\,dV = \int_B \left\{ (T_{ji}\,\mathring{u}_i)_{,j} + (\dot{\omega}_{i,j}\,M_{ji} + \dot{\omega}_i\,M_{ji,j}) - h_{i,i} \right\} dV$$

$$+ \int_B \left\{ \rho b_i \mathring{u}_i + \dot{\omega}_i\,\rho l_i + \rho q \right\} dV$$

or $\int_B \left\{ \mathring{u}_i\,(\rho a_i - T_{ji,j} - \rho b_i) + \rho\dot{\varepsilon} \right\} dV$

$$= \int_B \left\{ T_{ji}\,\mathring{u}_{i,j} + M_{ji}\,\dot{\omega}_{i,j} - h_{i,i} + \rho q + \dot{\omega}_i\,(M_{ji,j} + \rho l_i) \right\} dV$$

Write $T_{ji}\,\mathring{u}_{i,j} = [T_{ji}^S + T_{ji}^A][\frac{1}{2}(\mathring{u}_{i,j} + \mathring{u}_{j,i}) + \frac{1}{2}(\mathring{u}_{i,j} - \mathring{u}_{j,i})]$

$$= T_{ji}^S\,\dot{\epsilon}_{ij} - T_{ji}^A\,\dot{W}_{ij}$$

Also $\dot{\omega}_i\,(M_{ji,j} + \rho l_i) = -\dot{\omega}_i\,e_{ijk}\,T_{jk} = -T_{ji}\,\dot{W}_{ji} \equiv T_{ji}^A\,\dot{W}_{ij}$

T_{ij}^S and T_{ij}^A are symmetric and antisymmetric parts of stress tensor respectively.

Write $\dot{\omega}_{i,j} = \dot{k}_{ji}$ the curvature twist rate tensor

Then $M_{ji}\,\dot{\omega}_{i,j} = \left(M^{D}_{ji} + \frac{1}{3} M_{rr}\,\delta_{ij}\right) \dot{k}_{ji}$

$= M^{D}_{ji}\,\dot{k}_{ji}$ as $M_{rr} = 0$

Thus energy equation becomes

$$\int_B \rho \dot{\varepsilon}\, dv = \int_B \left(T^{S}_{ji}\,\dot{\epsilon}_{ij} + M^{D}_{ij}\,\dot{k}_{ij} - h_{i,i} + \rho q\right) dV$$

which yields

$$\rho\dot{\varepsilon} = T^{S}_{ij}\,\dot{\epsilon}_{ij} + M^{D}_{ij}\,\dot{k}_{ij} - h_{i,i} + \rho q$$

REFERENCES

1. GURTIN and STERNBERG., On the Linear Theory of Visco-elasticity, Archive of Rat. Mech. and Anal., Vol. 11,1962.

2. GROSS, B., 1947, On Creep and Relaxation, J. of Appl. Phys., Vol. 181, p. 212.

3. BOLTZMANN, L., Zur Theorie der elastischer Nachwirkung, See Hunter S.C. Progress in Solid Mech., Vol. 1.

4. KOLSKY and SHI (1958), National Bureau of Standards, Unpublished report, Brown University (Washington), See Hunter S.C. Progress in Solid Mech., Vol. 1.

5. TRUISDELL and TOUPIN, "The Classical field Theories", Encyclopaedia of Physics (Hand buchder Physik), Vol. III, Part 1 Springer 1960.

6. TOUPIN, R.A., Elastic materials with Couple Stresses, Arch. Rational Mech., Anal., 385-414 (1962).

7. MINDLIN and TIERSTEN, "Effects of Couple-stresses in Linear Elasticity", Arch. Rat. Mech. Anal., 415-448(1962).

8. STOKES, VIJAY KUMAR, Couple Stresses in Fluids, Phys. Fluids Vol. 9, 9 (1966).

9. HUNTER, S.C., Visco-elastic Waves, Progress in Solid Mech., Vol. 1, 1960.

10. FREDRICKSON, Principles and Applications of Rheology Prentice Hall Int. Series.

11. BLAND, D.R., The Theory of Linear Visco-elasticity, Int. Series.

12. TURNER ALFREY, Jr., Mechanical Behaviour of High Polymers, Inter Science.

13. SPAIN, BARRY, Tensor Calculus, Oliver and Boyd.